JOURNAL OF ALGEBRA 13, 48–56 (1969)

Algebras over Nonlocal Hensel Rings. II*

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Communicated by D. Buchsbaum

Received August 1, 1968

INTRODUCTION

This paper contains several applications and generalizations of the results obtained in our previous work with the same title, and deals with problems of lifting idempotents and projective modules, and with the study of full matrix algebras, and endomorphism algebras of projective modules.

Let A be a commutative ring, and $\mathfrak{m}$ an ideal of A such that $(A, \mathfrak{m})$ is a Hensel pair (Def. 2.1), and let R be an A-algebra, integral over A and finite over its center. Under these assumptions we show first (Th. 2.2) that every idempotent of $R/\mathfrak{m}R$ is image of an idempotent of R. This was proved in [9] when R is either commutative, or finite and projective over A. The proof given here consists mainly in reducing the problem to the commutative case, via a sort of weak Cohen–Seidenberg theorem for non commutative finite overrings of a commutative ring (Prop. 1.6). All this is contained in the first two sections.

In Section 3 we apply Theorem 2.2 to show that, in the above situation, every countable family of orthogonal idempotents of $R/\mathfrak{m}R$ can be lifted to R (Th. 3.1), which implies, by a standard argument, that if $R/\mathfrak{m}R$ is a full matrix algebra over some ring, so is R (Th. 3.3).

In Section 4 we observe that projective finitely generated $(R/\mathfrak{m}R)$-modules can be lifted to projective finitely generated R-modules (Th. 4.1), and we apply this fact, in Section 5, to show that if R is finite and projective over A, and $R/\mathfrak{m}R$ is the endomorphism algebra of a projective $(A/\mathfrak{m})$-module, then

* This research was done during the academic year 1967–68, when the author was a visitor at Brandeis University, with financial support of the National Science Foundation, grant NSF GP-4028.

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R is the endomorphism algebra of a projective A-module. This implies that the canonical homomorphism of the Brauer group of A into the Brauer group of $A/\mathfrak{m}$ is injective (Prop. 5.7).

Section 6 in only an erratum to [9].

It should be noted that the results contained in Sections 2 to 5 have been proved by Azumaya in [3] for finite algebras over local Henselian rings; in this case Azumaya's results are often stronger than ours, as we will point out throughout.

1. In this section we prove (Prop. 1.6) a somewhat noncommutative version of the following well known consequence of the Cohen–Seidenberg theorem: *let A be a commutative ring, B a commutative overring of A integral over A, $\mathfrak{m}$ an ideal of A. Then $\mathfrak{m}B \cap A \subset \sqrt{\mathfrak{m}}$* (see. e.g. [8], p. 77, ex. 3).

We begin with some preliminary remarks on the Jacobson radical of a ring. All rings and algebras are associative and with 1, and algebras are unitary (as always in this paper).

LEMMA 1.1. *Let R be a ring and B a commutative subring of R. Suppose that R is integral over B. Then* $\operatorname{rad} B \supset \operatorname{rad} R \cap B$.

Proof. Let C be a maximal commutative subring of R containing B. Then C is integral over B, whence $\operatorname{rad} C \cap B = \operatorname{rad} B$ ([8], p. 39, Cor. 3). Thus we may suppose $B = C$.

Let $b \in \operatorname{rad} R \cap B$. Then $1 + b$ is invertible in R ([5], p. 67, Cor. 1), and $(1 + b)^{-1}$ commutes with every element of B, since b does. Then $(1 + b)^{-1}$ is in B by maximality, and $b \in \operatorname{rad} B$, and the conclusion follows.

Remark 1.2. It is false, in general, that under the assumptions of Lemma 1.1 one has $\operatorname{rad} B = \operatorname{rad} R \cap B$. As an example let k be a field, R the ring of 2×2 matrices over k, and

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $B = k[u]$. Then B is commutative, and $u \in \operatorname{rad} B$, since $u^2 = 0$. But it is well known that $\operatorname{rad} R = 0$ (see e.g. [10], p. 11, Th. 3).

DEFINITION 1.3. Let A be a commutative ring and R an A-algebra. We say that R is *quasifinite* (over A) if R is integral over A, and is finite over its center.

LEMMA 1.4. *Let A be a commutative ring, R a quasifinite A-algebra, and $\mathfrak{m} = \operatorname{rad} A$. Then $\mathfrak{m}R \subset \operatorname{rad} R$.*

Proof. Let $\mathfrak{n}$ be a left maximal ideal of R, and suppose $\mathfrak{m}R \not\subset \mathfrak{n}$. Then $\mathfrak{m}R + \mathfrak{n} = R$. Let C be the center of R. Then $\mathfrak{m}C \subset \operatorname{rad} C$ since C in integral over A. But R is a finite C-module, and hence, by Nakayama's lemma we have $\mathfrak{n} = R$, a contradiction. Therefore $\mathfrak{m}R \subset \mathfrak{n}$, and the proof is complete.

Remark 1.5. If, in Lemma 1.4, we assume only that A is a commutative subring of R, the conclusion is false, even when R is a finitely generated A-module, as the example of Remark 1.2 shows.

PROPOSITION 1.6. *Let A be a commutative ring, R a quasifinite A-algebra* (Def. 1.3), *B a commutative sub-A-algebra of R. Let $\mathfrak{m}$ be an ideal of A. Them* $\mathfrak{m}R \cap B \subset \sqrt{\mathfrak{m}B}$.

Proof. We may assume that $A \subset B \subset R$, and we have to prove that if $\mathfrak{p}$ is a prime ideal of B containing $\mathfrak{m}B$, then we have also

$$\mathfrak{p} \supset \mathfrak{m}R \cap B$$

Let $\mathfrak{q} = A \cap \mathfrak{p}$, and let $B_\mathfrak{q} = B \otimes_A A_\mathfrak{q}$, $R_\mathfrak{q} = R \otimes_A A_\mathfrak{q}$. By flatness of $A_\mathfrak{q}$ we have inclusions

$$A_\mathfrak{q} \subset B_\mathfrak{q} \subset R_\mathfrak{q}$$

and it is clear that $R_\mathfrak{q}$ is a quasifinite $A_\mathfrak{q}$-algebra.

Consider the commutative diagram of A-algebras

$$\begin{array}{ccc} B & \longrightarrow & R \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B_\mathfrak{q} & \longrightarrow & R_\mathfrak{q} \end{array}$$

where all the homomorphisms are canonical. We have

$$\begin{aligned} \mathfrak{m}R \cap B &\subset [\varphi^{-1}(\mathfrak{m}R_\mathfrak{q})] \cap B \\ &= \psi^{-1}[(\mathfrak{m}A_\mathfrak{q})\, R_\mathfrak{q} \cap B_\mathfrak{q}] \end{aligned}$$

and since $\mathfrak{m} \subset \mathfrak{p} \cap A = \mathfrak{q}$ we get

$$\mathfrak{m}R \cap B \subset \psi^{-1}[(\mathfrak{q}A_\mathfrak{q})\, R_\mathfrak{q} \cap B_\mathfrak{q}] \tag{1}$$

But $R_\mathfrak{q}$ is a quasifinite $A_\mathfrak{q}$-algebra, and hence, by Lemma 1.4 we have

$$(\mathfrak{q}A_\mathfrak{q})\, R_\mathfrak{q} \subset \operatorname{rad} R_\mathfrak{q},$$

and by applying Lemma 1.1 to $R_\mathfrak{q}$ and $B_\mathfrak{q}$ we get

$$(\mathfrak{q}A_\mathfrak{q})\, R_\mathfrak{q} \cap B_\mathfrak{q} \subset \operatorname{rad} B_\mathfrak{q}. \tag{2}$$

Combining (1) and (2) we have

$$\mathfrak{m}R \cap B \subset \psi^{-1}(\mathrm{rad}\, B_{\mathfrak{q}}). \tag{3}$$

But $B_{\mathfrak{q}}$ is integral over $A_{\mathfrak{q}}$, and $\mathfrak{p}B_{\mathfrak{q}} \cap A_{\mathfrak{q}} = \mathfrak{q}A_{\mathfrak{q}}$. Hence $\mathfrak{p}B_{\mathfrak{q}}$ is a maximal ideal of $B_{\mathfrak{q}}$ ([*8*], p. 36, Prop. 1), and then $\mathrm{rad}\, B_{\mathfrak{q}} \subset \mathfrak{p}B_{\mathfrak{q}}$. Then (3) gives

$$\mathfrak{m}R \cap B \subset \psi^{-1}(\mathfrak{p}B_{\mathfrak{q}}) = \mathfrak{p}$$

and this completes the proof.

2. In this section we prove a theorem on lifting idempotents (Th. 2.2), which generalizes several results contained in [*9*] (Th. 2.1, Lemma 2.5, Th. 4.6), and we compare it with similar theorems proved by Azumaya for finite algebras over local Henselian rings (Remarks 2.3 and 2.4).

Before stating Theorem 2.2 we recall the definition of a Hensel pair.

DEFINITION 2.1. Let A be a commutative ring, and $\mathfrak{m}$ an ideal of A. We say that $(A, \mathfrak{m})$ is an H-pair (Hensel pair) if for every monic polynomial $f(X) \in A[X]$ and every decomposition $\bar{f}(X) = \bar{g}_0(X)\, \bar{h}_0(X)$ (in $(A/\mathfrak{m})[X]$) with $g_0(X)$, $h_0(X)$ monic and coprime, there is a unique pair $h(X)$, $g(X)$ of monic polynomials in $A[X]$ such that $f(X) = g(X)\, h(X)$ and $\bar{g}(X) = \bar{g}_0(X)$, $\bar{h}(X) = \bar{h}_0(X)$. Moreover $g(X)$ and $h(X)$ are coprime.

The condition that $g(X)$ and $h(X)$ in the above definition are coprime (for every decomposition and every $f(X)$ as above) is equivalent to: $\mathfrak{m} \subset \mathrm{rad}\, A$. For more details see [*11*].

THEOREM 2.2. *Let* $(A, \mathfrak{m})$ *be an* H-*pair* (*Def.* 2.1) *and let* R *be a quasifinite* A-*algebra* (Def. 1.3). *Let* $u \in R$ *be such that* $\bar{u} \in R/\mathfrak{m}R$ *is idempotent. Then there is a polynomial* $f(X) \in A[X]$ *such that* $f(u) \in R$ *is idempotent, and* $\overline{f(u)} = \bar{u}$. *In particular every idempotent of* $R/\mathfrak{m}R$ *is image of an idempotent of* R.

Proof. Let $B = A[u] \subset R$. Then B is commutative and integral over A, and hence $(B, \mathfrak{m}B)$ is an H-pair ([*9*], Th. 4.6). Let $\mathfrak{n} = \mathfrak{m}R \cap B$. Then $\mathfrak{n} \subset \sqrt{\mathfrak{m}B}$ by Proposition 1.6, whence $(B, \mathfrak{n})$ is an H-pair ([*9*], Cor. 4.2). Then every idempotent of $B/\mathfrak{n}$ can be lifted to an idempotent of B. ([*9*], Prop. 1.7). Since the inclusion $B \to R$ induces an inclusion $B/\mathfrak{n} \to R/\mathfrak{m}$, it is clear that $\bar{u}$ is an idempotent of $B/\mathfrak{n}$. Then there is an idempotent $e \in B = A[u]$ such that $\bar{e} = \bar{u}$, and the conclusion is clear.

Remark 2.3. For a local Henselian ring A, Azumaya proved the following theorem: *let* R *be a finite* A-*algebra, and* $\mathfrak{b}$ *a twosided ideal of* R. *Then every idempotent of* $R/\mathfrak{b}$ *is image of an idempotent of* R. (see [*3*], Th. 22, or [*7*], p. 127, Ex. 5). A reasonable generalization of Azumaya's theorem and

Theorem 2.2 above seems the following: *Let A be a commutative ring, R a quasifinite A-algebra, $\bar{A}$ the image of A in R, $\mathfrak{b}$ a twosided ideal of R, $\mathfrak{m} = \mathfrak{b} \cap \bar{A}$. If $(\bar{A}, \mathfrak{m})$ is an* H*-pair, then every idempotent of $R/\mathfrak{b}$ is image of an idempotent of R.* This conjecture is justified since its assumption is verified in both Azumaya's theorem and ours, as follows easily by [*9*], Corollary 4.2.

Remark 2.4. I don't know whether Theorem 2.2 is true in general under the weaker assumption "R integral over A", not even when A is local. Our proof of Theorem 2.2 doesn't seem to extend to this case, unless one is able to get a suitable generalisation of Lemma 1.4.

3. In this section we apply Theorem 2.2 to the problem of lifting families of orthogonal idempotents (Th. 3.1 and Cor. 3.2), and obtain a criterion for an algebra to be a full matrix algebra (Th. 3.3). All these results are similar to several known theorems (see e.g. [*3*], Th. 24 and 25; [*5*] p. 74, Ex. 10; [*7*], p. 127, Ex. 5; [*10*], Chap. III, Prop. 4 and 5, and Th. 1).

THEOREM 3.1. *Let $(A, \mathfrak{m})$ be an* H*-pair (Def. 2.1) and let R be a quasifinite A-algebra (Def. 1.3). Let $u_1, u_2, \ldots$ be a sequence of elements of R such that $\bar{u}_i\bar{u}_j = \delta_{ij}\bar{u}_j$ (in $R/\mathfrak{m}R$) for all i, j. Then there is a sequence $e_1, e_2, \ldots$ of elements of R such that $e_ie_j = \delta_{ij}e_j$ for all i, j, and $\bar{e}_i = \bar{u}_i$ for all i.*

Proof. We proceed by induction. It is clear that e_1 exists by Theorem 2.2. Therefore we may assume that there are $e_1, \ldots, e_{n-1} \in R$ such that $e_ie_j = \delta_{ij}e_j$ and $\bar{e}_i = \bar{u}_i$ $(0 < i, j \leqslant n-1)$, and we construct e_n as follows.

Let $e = \sum_1^{n-1} e_i$ and put

$$u = u_n - eu_n - u_ne \tag{1}$$

Then $\bar{u} = \bar{u}_n$, and hence, by Theorem 2.2, there is a polynomial $f(X) \in A[X]$ such that $f(u)$ is idempotent, and $\overline{f(u)} = \bar{u} = \bar{u}_n$. Let $e_n = f(u)$.

To complete the proof we have to show that $e_ne_i = e_ie_n = 0$ for all $i < n$. It is clear that e is idempotent, and then it follows easily by (1) that $eu = ue$, whence $e_ne = ee_n$, since e_n is a polynomial in u with coefficients in A. Then e_ne is idempotent, and since $\bar{e}_n\bar{e} = \bar{u}_n \sum_1^{n-1} \bar{u}_i = 0$, we have $e_ne \in \mathfrak{m}R \subset \operatorname{rad} R$ (Lemma 1.4). Then $1 - e_ne$ is invertible and idempotent at the same time, whence

$$e_ne = ee_n = 0 \tag{2}$$

Now if $i < n$, by the induction hypothesis we have $ee_i = e_ie = e_i$, whence, by (2),

$$e_ne_i = e_nee_i = 0$$
$$e_ie_n = e_iee_n = 0$$

and this completes the proof.

COROLLARY 3.2. *Let $(A, \mathfrak{m})$ be an* H*-pair, and R a quasifinite A-algebra. Let $u_1, \dots, u_n \in R$ be such that $\bar{u}_i\bar{u}_j = \delta_{ij}u_j$ $(0 < i, j \leqslant n)$ and $\sum_1^n \bar{u}_i = 1$ (in $R/\mathfrak{m}R$). Then there are $e_1, \dots, e_n \in R$ such that $e_ie_j = \delta_{ij}e_j$, $\bar{e}_i = \bar{u}_i$ $(0 < i, j \leqslant n)$ and $\sum_1^n e_i = 1$.*

Proof. By Theorem 3.1 there are $e_1, \dots, e_n \in R$ such that $e_ie_j = \delta_{ij}e_j$ and $\bar{u}_i = \bar{e}_i$ $(0 < i, j \leqslant n)$. Let $e = \sum_1^n e_i$. Then e is idempotent, and $\bar{e} = \sum_1^n \bar{u}_i = 1$. But $\mathfrak{m}R \subset \operatorname{rad} R$ (Lemma 1.4), and thus e is invertible. Then it must be equal to 1, and the proof is complete.

THEOREM 3.3. *Let $(A, \mathfrak{m})$ be an* H*-pair, and let R be a finite A-algebra. Suppose there is a finite $(A/\mathfrak{m})$-algebra $\bar{S}$ such that $R/\mathfrak{m}R \simeq M_n(\bar{S})$ (the full matrix algebra of rank n over $\bar{S}$). Then there is a finite A-algebra S such that $\bar{S} = S/\mathfrak{m}S$ and $R \simeq M_n(S)$.*

Proof. It follows by the above corollary with the same argument used in [*10*] to prove Theorem 1 on page 55.

Remark 3.4. If one regards $M_n(\bar{S})$ as the endomorphism algebra of a free $\bar{S}$-module, the following conjecture appears as a reasonable generalization of Theorem 3.3: *Suppose $(A, \mathfrak{m})$ and R are as in Theorem* 3.3, *and suppose there are an $(A/\mathfrak{m})$-algebra $\bar{S}$ and a projective left $\bar{S}$-module $\bar{E}$ such that $R/\mathfrak{m}R \simeq \operatorname{End}_{\bar{S}}(\bar{E})$. Then there are an A-algebra S and a projective left S-module E such that $\bar{S} \simeq S/\mathfrak{m}S$, $\bar{E} \simeq E/\mathfrak{m}E$ and $R \simeq \operatorname{End}_S(E)$.*

In Section 5 the above conjecture is proved in the special case $\bar{S} = A/\mathfrak{m}$ (Th. 5.5).

Remark 3.5. In Theorem 3.3 it is not always true that if $\bar{S} = A/\mathfrak{m}$ then $S = A$, as the trivial example $R = \bar{R} = A/\mathfrak{m}$ shows. However this is the case when R is projective, as we will prove later (Corollary 5.6).

4. It is easy to see that Theorem 2.2 can be applied to lift projective modules, thus generalizing some results obtained in [*9*] (Th. 5.1, Cor. 5.2 and 5.3), without changing the proofs. As an example we state a generalization of [*9*], Theorem 5.1. In the following, "module" always means "finitely generated unitary left module".

THEOREM 4.1. *Let $(A, \mathfrak{m})$ be an* H*-pair (Def.* 2.1) *and R a quasifinite A-algebra (Def.* 1.3)*. Let $\bar{E}$ be a projective $R/\mathfrak{m}R$-module. Then there is a unique (up to isomorphism) projective R-module E such that $E/\mathfrak{m}E \simeq \bar{E}$.*

The following corollary is immediate, and is a partial generalization of [*4*], Lemma 18.1.

COROLLARY 4.2. *Let* $(A, \mathfrak{m})$ *and* R *be as in Theorem* 4.1. *Then the canonical homomorphism* $K^0(R) \to K^0(R/\mathfrak{m}R)$ *is bijective* (here $K^0(S)$, S any ring, denotes the Grothendieck group of the category of projective S-modules; for more more details see e.g. [4] Chap. III, or [6], p. 178, Ex. 10).

5. In this section we apply Theorem 4.1 to endomorphism algebras of projective modules (Th. 5.5) and to the Brauer group of a commutative ring (Prop. 5.7).

Before proving Theorem 5.5 we recall several properties of projective modules. As before all modules are supposed to be left finitely generated unitary modules.

LEMMA 5.1. *Let* B *be a commutative ring, and let* E *be a projective* B*-module. Then the following conditions are equivalent*

(i) E *is faithful* (i.e. $\mathrm{Ann}(E) = 0$)

(ii) $\mathrm{Supp}(E) = \mathrm{Spec}(B)$

(iii) *For every maximal ideal* $\mathfrak{n}$ *of* B, $E \otimes B_{\mathfrak{n}} \neq 0$.

Proof. Since $\mathrm{Supp}(E) = V(\mathrm{Ann}(E))$ ([6], p. 132, Prop. 17) it is clear that (i) $\Rightarrow$ (ii). Obviously (ii) $\Rightarrow$ (iii). In order to prove that (iii) $\Rightarrow$ (i) we observe first that if $\mathfrak{b} = \mathrm{Ann}(E)$ we have $\mathfrak{b}A_{\mathfrak{n}} = \mathrm{Ann}(E \otimes B_{\mathfrak{n}})$ for all maximal ideals $\mathfrak{n}$ of B. ([6], p. 29, formula (9)). Since $E \otimes B_{\mathfrak{n}}$ is a nonzero free $B_{\mathfrak{n}}$-module it follows that $\mathfrak{b}B_{\mathfrak{n}} = 0$ for all maximal ideals $\mathfrak{n}$ of B, and then $\mathfrak{b} = 0$ ([6], p. 112, Cor. 2).

Remark 5.2. Lemma 5.1 is false without the assumption "E projective". Take for instance B a local nonreduced ring, and $E = B/\mathfrak{b}$, where $\mathfrak{b}$ is the nilradical of B. Then $\mathrm{Ann}(E) = \mathfrak{b} \neq 0$, but $\mathrm{Supp}(E) = V(\mathfrak{b}) = \mathrm{Spec}(B)$.

LEMMA 5.3. *Let* B *be a commutative ring,* $\mathfrak{m}$ *an ideal of* B *contained in* $\mathrm{rad}(B)$, E *a projective* B*-module. Then the following conditions are equivalent*:

(i) E *is a faithful* B*-module*

(ii) $E/\mathfrak{m}E$ *is a faithful* $(B/\mathfrak{m})$*-module.*

Proof. (ii) $\Rightarrow$ (i) by Lemma 5.1 (iii), and the converse follows easily by the same lemma and Nakayama's lemma.

LEMMA 5.4. *Let* B *be a commutative ring, and let* E *be a projective* B*-module. Let* $R = \mathrm{End}_B(E)$. *Then, as a left* R*-module,* E *is projective.*

Proof. See e.g. [1], Proposition A3.

Now we prove

THEOREM 5.5. *Let* $(A, \mathfrak{m})$ *be an* H-*pair* (*Def.* 2.1) *and* R *a finite projective* A-*algebra. Suppose there is a projective* $(A/\mathfrak{m})$-*module* $\bar{E}$ *such that* $R/\mathfrak{m}R \simeq \mathrm{End}_{A/\mathfrak{m}}(\bar{E})$. (*as* $(A/\mathfrak{m})$-*algebras*). *Then there is a unique* (*up to isomorphism*) *projective* A-*module* E *such that* $E/\mathfrak{m}E \simeq \bar{E}$ *and* $R \simeq \mathrm{End}_A(E)$ (*over* A). *Moreover* E *is a faithful* A-*module if and only if* $\bar{E}$ *is a faithful* $(A/\mathfrak{m})$-*module.*

Proof. By Lemma 5.4 E is a projective $(R/\mathfrak{m}R)$-module. Hence by Theorem 4.1 (or else by [*9*], Th. 5.1) there is a projective R-module E such that $E/\mathfrak{m}E \simeq \bar{E}$.

Every element $r \in R$ gives an A-endomorphism f_r of E, defined by $f_r(x) = rx$ for all $x \in E$. It is easy to see that the map $\varphi : r \to f_r$ is a homomorphism $\varphi : R \to \mathrm{End}_A(E) = R'$ (of A-algebras).

Since E is R-projective and R is A-projective, one sees that E is A-projective; by then it is easy to see that φ induces an isomorphism of $(A/\mathfrak{m})$-algebras $\bar{\varphi} : R/\mathfrak{m}R \to R'/\mathfrak{m}R'$. Furthermore R' is a projective A-module and since $\mathfrak{m} \subset \mathrm{rad}\, A$ ([*9*], Lemma 1.6) it follows, by a standard argument involving Nakayama's lemma, that φ is bijective. This proves the former statement. The latter follows at once by Lemma 4.3.

COROLLARY 5.6. *Let* $(A, \mathfrak{m})$ *be an* H-*pair and* R *a finite projective* A-*algebra. If* $R/\mathfrak{m}R = M_n(A/\mathfrak{m})$, *then* $R \simeq M_n(A)$

Proof. It follows by Theorem 5.5 applied to $\bar{E} = (A/\mathfrak{m})^n$.

Theorem 5.5 may be interpreted in terms of the Brauer group, as shown in Proposition 5.7, which is a generalization of a similar result for local rings ([*2*], Prop. 6.1).

For a commutative ring B we denote by $\mathscr{B}(B)$ the Brauer group of B, as defined by Auslander and Goldman in [*2*], Section 5. We refer the reader to [*2*] for details.

PROPOSITION 5.7. *Let* $(A, \mathfrak{m})$ *be an* H-*pair. Then the canonical homomorphism* $\mathscr{B}(A) \to \mathscr{B}(A/\mathfrak{m})$ *is injective.*

Proof. By [*2*], Proposition 5.3 we have to show the following: *Let* R *be a central separable* A-*algebra* ([*2*], *Section* 1), *and suppose that*

$$R/\mathfrak{m}R \simeq \mathrm{End}_{A/\mathfrak{m}}(\bar{E})$$

where $\bar{E}$ *is a faithful projective* $(A/\mathfrak{m})$-*module. Then* $R \simeq \mathrm{End}_A(E)$ *where* E *is a faithful projective* A-*module.*

Now a central separable A-algebra is finite and projective over A ([*2*], Th. 2.1 c)) and thus the conclusion follows at once by Theorem 5.5.

Remark 5.8. Azumaya proved that $\mathscr{B}(A) \simeq \mathscr{B}(A/\mathfrak{m})$ whenever A is a local Henselian ring with maximal ideal $\mathfrak{m}$ ([*3*], Th. 31). A proof of this fact when A is a complete noetherian local ring appears in [*2*] (Th. 6.5). The problem for (nonlocal) Hensel pairs, as far as I know, is still open, even for $\mathfrak{m}$-adically complete and separated noetherian rings.

6. This section is an erratum to [*9*].

(6.1) In the statement of Corollary 1.3 i), replace $(A, \mathfrak{m})$ by $(A, \mathfrak{n})$.

(6.2) Definition 1.5 is not precise, and is to be replaced by Definition 2.1 of the present paper. For better details see [*11*].

(6.3) The last two lines of the proof of Theorem 2.1 are not correct. Replace them by the following: "Let $\sigma : S \to R$ be the canonical homomorphism. Then $\sigma(H(u))$ is an idempotent lifting $\bar{u}$"

(6.4) In the statement of Theorem 6.8 (ii) replace "free" by "projective". The proof remains unchanged. I don't know whether Theorem 6.8 remains true in its old formulation.

ACKNOWLEDGMENT

The author wishes to thank Dr. Mark Ramras of Harvard University for several useful conversations on the subject of this paper.

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